The Navier–Stokes Proof, Step by Step: What OpenAI Claimed, What the Machines Checked, and Where the Argument Breaks

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repository: https://github.com/truemagic-coder/ns-proof

Abstract

This is a step-by-step walkthrough of the argument in the OpenAI manuscript "Finite Time Blowup for Navier–Stokes" – the claim, the construction, and the closing argument of §10.3 – with a machine-verdict attached to every step where a machine can decide it. Each part has two lanes: a main lane in plain language, and a technical lane with the actual estimates. The walk ends where the argument breaks: the pressure-flux estimate (10.19), whose derivation rests on three claims that are each false for every admissible choice of cutoff, and whose bound is false under its own quantitative hypotheses – with a counterexample family whose pressure flux is nonzero while the right-hand side of the estimate is exactly zero. The companion counterexample paper [**?**] carries the full construction; this paper carries the walk.

Part I
The claim

The paper's claim is a single sentence, and it is worth reading slowly.

manuscript p. 1 #2

Theorem 1.1. For every $\nu > 0$ there exist a force $f \in C_c^\infty(\mathbb{R}^3 \times (0,\infty); \mathbb{R}^3)$, a compact set $K \subset \mathbb{R}^3$, and smooth velocity and pressure fields u, p on $\mathbb{R}^3 \times [0,1)$ satisfying

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f, \qquad \nabla \cdot u = 0, \qquad u(\cdot, 0) = 0, \tag{1.1}$$

such that $\operatorname{supp} u(\cdot,t) \cup \operatorname{supp} p(\cdot,t) \subset K$ for every $0 \le t < 1$,

$$\sup_{0 \le t < 1} \|u(t)\|_{L^2(\mathbb{R}^3)} < \infty, \qquad \limsup_{t \uparrow 1} \|u(t)\|_{L^\infty(\mathbb{R}^3)} = \infty.$$

Main lane. For any viscosity, the paper constructs a force (smooth, compactly supported) and a solution that starts at rest, stays bounded in energy, and blows up in velocity as $t \uparrow 1$. The constructed solution *does not* solve the equations up to time 1 – that is the point. The claim that

matters for the Millennium problem is the second half of §10.3: *no* smooth solution with the same force, zero initial data, and bounded energy can exist either.

Technical lane — the real math

The non-existence clause: the constructed u satisfies the equations on $[0,1)$, $\sup_{t<1}\|u(t)\|_2 \le E_0 < \infty$, and $\limsup_{t\uparrow 1}\|u(t)\|_\infty = \infty$ (equation (10.21) of the manuscript). If any smooth solution v with the same data and $\sup_t \|v(t)\|_2 < \infty$ existed, the comparison argument of §10.3 gives $v = u$ on every $[0,T]$, $T < 1$ – hence on $[0,1)$ – contradicting (10.21). That is exactly alternative (C) in Fefferman's formulation [1]: bounded energy is assumed, and existence is denied.

Checkpoint 1 `cp-claim`

Evidence: manuscript p.1 (Theorem 1.1, quotes #q-thm11, #q-noclause in [?]); the comparison-lemma consumption is p.123

Main lane, continued. The construction itself – a blow-up solution built on an auxiliary torus and glued into $\mathbb{R}^3$ – is the first 90 pages. We do not walk it here: it is the part the machines *can* and did check, and the walk that matters for the verdict is the closing argument, §10.3, where the paper proves no smooth solution can avoid the blowup.

Technical lane — the real math

The construction's output, as used by §10.3: a solution u on $\mathbb{R}^3 \times [0,1)$ with (i) the equations (1.1), (ii) $\sup_{t<1}\|u(t)\|_2 \le E_0$, (iii) $\limsup_{t\uparrow 1}\|u(t)\|_\infty = \infty$ (eq. (10.21)), (iv) uniform bounds on ∇u in space-time compacts of $[0,1)$ that feed the constants C_T of §10.3. The §10.3 argument consumes only (i)–(iv) and the force.

Part II

The argument, step by step

The closing argument has one shape: compare the constructed solution u with any putative smooth solution v, and show the difference $w = v - u$ must be zero. The difference is not a solution of the same equations (the equations are nonlinear), so the argument bounds the *energy of the difference* on an expanding ball and pushes the ball radius to infinity.

Technical lane — the real math

For $R \ge 1$ and a smooth cutoff ϕ (equal to 1 on the unit ball, supported in B_c), write $\chi_R = \phi_{8R}$ and

$$E_R(t) = \left(\int \chi_R |w|^2\right)^{1/2}, \qquad A_R(t) = \left(\int \chi_R |\nabla w|^2\right)^{1/2}, \qquad B_R(t) = \|\phi_{4R} w\|_6.$$

The difference-energy identity (after subtracting the two equations, integrating by parts, and recovering the pressure difference $\pi = P - p$ via the Riesz transforms, eq. (10.16)) closes to

$$\tfrac{1}{2}E_R' + \tfrac{1}{2}A_R^2 \le \|\nabla u\|_\infty E_R + \text{flux terms},$$

where the flux terms come from $\int \pi w \cdot \nabla\chi_R \, dx$ through the annulus where $\nabla\chi_R \ne 0$. The two flux sums are (10.18).

Step 1 – the pressure recovery. The pressure difference is recovered from the velocity difference by the Riesz transforms. This is mechanical, and the manuscript is explicit about it.

Technical lane — the real math

$\pi = P - p = \sum_{i,j} R_i R_j (v_i v_j - u_i u_j)$ (eq. (10.16), with R_i the Riesz transforms). The flux $\int \pi w \cdot \nabla \chi_R \, dx$ splits into two sums (10.18): one pairing the $L^{3/2}$-small first sum with $CR^{-1}\phi_{3R}|w|$, and one pairing the commutator against $\nabla\chi_R/R$-type kernels.

Checkpoint 2 `cp-structure`

Evidence: manuscript p.122 (eqs. (10.16), (10.17), (10.18); quotes #q-setup, #q-firstsum in [?])

Step 2 – the first flux sum. One sum in (10.18) is bounded in $L^{3/2}$ by $C_T(B_R + 1)$: the $w \otimes w$ part via $B_R\|w\|_2$, the $w \otimes u$ part via $\|w\|_2\|u\|_6$.

Technical lane — the real math

$$\|\phi_{4R} w_i w_j\|_{3/2} \le B_R \|w\|_2, \qquad \|\phi_{4R} w_i u_j\|_{3/2} \le \|w\|_2 \|u\|_6 \qquad \Rightarrow \qquad \|\cdot\|_{3/2} \le C_T(B_R + 1),$$

using $\sup_t \|w(t)\|_2 \le M_0$ and $\sup_t \|u(t)\|_6 \le U_0$ (the latter from Sobolev on the bounded energy). This step is correct as stated.

Step 3 – the commutator. The second sum is a commutator of the Riesz kernel with the cutoff; it is bounded in $L^{4/3}$ by $C_T R^{-3/4}$ via Young's convolution inequality.

Technical lane — the real math

The kernel is $K_R(x - y) = C|x - y|^{-3} \min(|x - y|/R, 1)$; its $L^{4/3}$-type integral is $\int_0^R r^{-2/3} dr / R^{4/3} + \int_R^\infty r^{-2} dr \le CR^{-1}$, and Young gives the $L^{4/3}$ norm of the convolution as $C_T R^{-3/4}$ (the $R^{-3/4} = (R^{-1})^{3/4}$ interpolation of the L^1/L^∞ halves). This step is also correct as stated.

Checkpoint 3 `cp-kernels`

Evidence: manuscript p.122–123 (quotes #q-firstsum, #q-young in [?]); the kernel computation re-derived in [?], §3.4

Step 4 – the pairing, and (10.19). Here the argument makes the move that does not go through. The paper claims the two flux terms pair with $CR^{-1}\phi_{3R}|w|$, then interpolates $\phi_{3R} w$ between L^2 and L^6 to get the bound in terms of B_R and $\|w\|_2$.

Technical lane — the real math

The claimed pairing and its consequences (manuscript p.123, quote #q-pairing in [?]):

- "$|\nabla\chi_R| \le CR^{-1}\phi_{7R}$" – **false**: $\operatorname{supp}\nabla\chi_R \subset [8R, 8cR]$ while $\phi_{7R} \equiv 0$ on $(7cR, \infty)$; the shell $(7cR, 8cR]$ is nonempty;
- "pair with $CR^{-1}\phi_{3R}|w|$" – **impossible**: $\operatorname{supp}\phi_{3R} \subset B_{3cR} \subset B_{8R}$, disjoint from the flux support $[8R, 8cR]$;
- "$\|\phi_{3R}w\|_3 \le \|\phi_{2R}w\|_3$" – **reversed** for the standard cutoff (and the second link to $B_R^{1/2}\|w\|_2^{1/2}$ is the only one that holds).

The conclusion the paper draws – (10.19), the flux bounded by $C_T R^{-1}(B_R+1)B_R^{1/2} + R^{-3/4}B_R^{3/4}$ – does not follow.

Checkpoint 4 `cp-defect`

Evidence: manuscript p.123 (quote #q-pairing in [?]); the three false claims in [?], §2.3 – each false for every *admissible cutoff*

Step 5 – the closure (as claimed). Using (10.17) to replace every A_R by $B_R + R^{-1}\|w\|_2$, every power of A_R in the flux bounds would be at most $3/2 < 2$, and Young's inequality would absorb the A_R terms into the dissipation:

Technical lane — the real math

Claimed closure (manuscript p.123, quote #q-using17 in [?]):

$$\tfrac{1}{2}E_R' + \tfrac{1}{2}A_R^2 \le \|\nabla u\|_\infty E_R + \frac{C_T}{2R},$$

after which Gronwall on E_R, the $R \to \infty$ limit (using $\chi_R \uparrow 1$), and the difference-energy identity give $w \equiv 0$ on $[0, T]$ for every $T < 1$ – Lemma 10.5.

Step 6 – the contradiction. With $v = u$ on $[0, 1)$ and (10.21), no smooth bounded-energy solution with the same data exists. The theorem is claimed.

Technical lane — the real math

Lemma 10.5 (the comparison lemma): any smooth solution v with the same force, $v(\cdot, 0) = 0$, and $\sup_{t\ge 0}\|v(t)\|_2 < \infty$ satisfies $v = u$ on $\mathbb{R}^3 \times [0, 1)$ – contradicting $\limsup_{t\uparrow 1}\|u(t)\|_\infty = \infty$ (eq. (10.21)). Hence Theorem 1.1's non-existence clause, hence alternative (C) [1]. *The chain's load-bearing estimate is (10.19) – the step that fails.*

Checkpoint 5 `cp-chain`

Evidence: manuscript p.123–124 (Lemma 10.5 and the closure; quote #q-using17 in [?]); the Lean-side chain in `NavierStokes/R3/WholeSpaceUniqueness.lean:30--104`

Part III

Where the argument breaks

The failure is not in the kernel estimates, not in the pressure recovery, and not in Gronwall. It is in the *support bookkeeping* of the flux pairing: the estimate the paper needs is about a quantity on the annulus $[8R, 8cR]$, and the quantities it proposes to control it with live strictly inside the ball.

Main lane. Think of the geometry. The flux $\int \pi w \cdot \nabla\chi_R \, dx$ measures what leaks through the boundary of the ball of radius $8R$ – the integrand is zero everywhere except the shell between $8R$ and $8cR$. But $B_R = \|\phi_{4R} w\|_6$ is an average over the much smaller ball B_{4cR}, and $\|w\|_2$ is a global average. A flux through a remote shell cannot be bounded by averages over the interior – and the counterexample makes that precise.

Technical lane — the real math

Counterexample (full construction in [?], §3). For every $\varepsilon > 0$: smooth divergence-free u, $v_\varepsilon = u + \varepsilon\psi$ with $\operatorname{supp}\psi \subset (8, 16)$, $w_\varepsilon = \varepsilon\psi$, $\pi_\varepsilon = \sum R_i R_j g_{ij}$ with $g_\varepsilon = v_\varepsilon v_\varepsilon - uu$. Then $B_{1,\varepsilon} = \|\phi_4 w_\varepsilon\|_6 = 0$ *exactly* (the support of ψ starts at 8, where ϕ_4 is already zero), all the quantitative hypotheses of the derivation hold ($\|w_\varepsilon\|_2 = O(\varepsilon)$, $\sum \operatorname{tr} \|g_\varepsilon\|_1 = O(\varepsilon)$), and yet, by FFT Riesz transforms on a 64^3 grid:

$$\left|\int \pi_\varepsilon w_\varepsilon \cdot \nabla\chi_1 \, dx\right| = \begin{cases} -1.652947e-02\,\varepsilon^2 & \varepsilon = 1 \\ -4.132369e-03\,\varepsilon^2 & \varepsilon = 0.5 \ \ (\text{ratio } 0.2500 = 0.2500) \\ -1.652947e-04\,\varepsilon^2 & \varepsilon = 0.1 \ \ (\text{ratio } 0.0100 = 0.0100) \end{cases} > 0 = \text{RHS of (10.19), for all } C_T.$$

The exact ε^2 scaling is structural (the flux is bilinear in w_ε through the cross terms), not a fit.

Checkpoint 6 `cp-cf`

Evidence: `counterexample/counterexample-1019.py` *(numpy only; regenerates all numbers above from* `counterexample/results.json`*); the paper's numbers are generated, not hand-typed – [?], Appendix A*

Main lane, the direction of the dependency. What does the right bound look like? The fork's own Lean code – the object the verification job actually verified – proves the flux bound with the *dissipation* A_R in it:

Technical lane — the real math

Lean (verified; `NavierStokes/R3/PressureFlux.lean:584`):

$$\left|\int \pi w \cdot \nabla\chi_R\right| \le C_T(B_R^{1/2} + 1)\left(\frac{A_R}{R} + \frac{1}{R^2}\right) + R^{-7/4} B_R^{3/4}.$$

This is exactly "the weighted dissipation inside the ball" the paper's own purpose sentence promises (quote #q-purpose in [?]) – and it is the type of bound the geometry requires: the flux through $[8R, 8cR]$ is controlled by what happens *inside*, through the dissipation, not by the L^6 average on a smaller ball. Note the arrow: (10.17) gives $B_R \le C(A_R + R^{-1}\|w\|_2)$, so B_R is controlled *by* A_R – the other direction, which a B_R-only flux bound would need, does not exist.

Checkpoint 7 `cp-arrow`

Evidence: fork `NavierStokes/R3/PressureFlux.lean:584` *(comparator-checked, [2], mapping table row* `10.3-flux`*); the direction argument in [?], §4*

Main lane, the honest scope. In the paper's argument $w = v - u$ is the difference of two *solutions*; the counterexample uses general smooth divergence-free fields. It proves (10.19) is not a consequence of the quantitative hypotheses the derivation invokes – the derivation is invalid and the estimate is false, not merely unproved. Whether a B_R-only bound holds for solution-differences is an open question. The published derivation does not establish it, and the Lean code does not prove it either.

Part IV
What remains, and what it means

Main lane. Three things are settled, and one thing is open.

1. *Settled: the page's (8.19) is sound.* The other flagged divergence – the auxiliary-torus inverse estimate – we formalized in Lean, and the kernel accepted the paper's argument as written: r torus derivatives of the inverse are bounded by $r{+}4$ input derivatives, via the summable $(1{+}|k|)^{-3}$ majorant and the symmetric $(1{+}|n|)^{-3/2}$ split. The Lean code proves a stronger $m{+}5$ version; the page's $m{+}4$ version also holds.

2. *Settled: the code is a different argument.* The verified Lean code proves the A_R-dependent flux bound, not the page's B_R-only one. A fully green verdict verifies the code, not the paper's argument.

3. *Settled: the repair's ingredients hold.* The corrected estimates $(10.19)'$ and $(10.17)'$ – the flux bound with the L^6 norm measured on a ball that covers the annulus, and the matching Sobolev inequality – are formalized in the fork and kernel-verified.

4. *Open: the closure.* Whether the corrected flux bound closes the difference-energy inequality and the full Gronwall/$R \to \infty$ chain is not yet formalized. That is the next step, and until it is done, "the repaired argument works" is a reading, not a verdict.

Technical lane — the real math

The four items as machine-verdicts:

- (8.19): `NavierStokes/SmoothFamilyTorusInverse.lean:1109` (`norm_derivativeWord_inverse_le_m4`, fork `9bbfa7b`) – kernel-accepted, single attempt;
- the code's argument: mapping table rows `10.3-flux` vs. the page's (10.19) – [2], §3;
- the repair: `NavierStokes/R3/Paper1019Repaired.lean:187--240` (fork `24b1944`) – kernel-accepted;
- the closure: open row of the mapping table – not a verdict.

Checkpoint 8 `cp-819`

Evidence: fork `9bbfa7b` (`NavierStokes/SmoothFamilyTorusInverse.lean:1109`); the full formalization in [2], §2

Checkpoint 9 `cp-code`

Evidence: [2] mapping table (16 rows, each machine-checked against the fork at file/line/commit); the signed verification report (job `31974058-a665-472d-b6f8-00e4ec9e2d45`)

Checkpoint 10 `cp-repair`

Evidence: fork `24b1944` (`NavierStokes/R3/Paper1019Repaired.lean:187--240`); status and the open closure row in [2], §4–5

Main lane, the lesson. A green verification verdict is a specific claim: the Lean object proves itself, under the permitted axioms, re-checked by a second kernel. It is not a claim about the page. The gap between the two – the alignment question – has to be worked, symbol by symbol, with the kernel as referee for what is formalizable and a counterexample for what is not. That is exactly what this series is: [**?**] (the counterexample and the defect), this paper (the walk), [2] (the verified object and the mapping), [3] (the code itself, in detail).

References

[1] Clay Mathematics Institute. The Millennium prize problems: Navier–Stokes existence and smoothness. URL https://www.claymath.org/millennium-problems/navier-stokes-equations.

[2] Bevan Hunt. The proof as the machine checked it: The verified Lean formalization of the navier–stokes blowup argument. Companion paper, this series, 2026. DOI: PENDING.

[3] Bevan Hunt. Inside the verified proof: A step-by-step walkthrough of the Lean navier–stokes formalization. Companion paper, this series, 2026. DOI: PENDING.